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# Option Pricing with Periodic Volatility: A Modified Black-Scholes Model using Jacobi Elliptic Functions

Chinwe N. Obi [1], Bright O. Osu [2], Promise A. Azor [3,*]
[1] Department of Mathematics Federal University of Technology Owerri, Nigeria
[2] Department of Mathematics Abia State University, Uturu, Nigeria
[3] Department of Mathematics and Statistics Federal University Otuoke, Nigeria

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**ABSTRACT**

In this work, a modified Black-Scholes model for European option pricing is presented, in which the volatility function exhibits a periodic pattern determined by the Jacobi elliptic sine function. This method maintains smoothness and mathematical tractability while capturing organized, oscillatory activity frequently seen in financial markets.

We develop the relevant parabolic partial differential equation and confirm regularity and uniform ellipticity criteria to confirm that it is well-posed. The complete model is solved nu-merically using a Crank-Nicolson finite difference method after a formal solution is derived using the Fourier transform under some assumptions. The findings of the simulation show how volatility patterns and option prices are impacted by changes in the elliptic modulus. Specifi-cally, for short-term and at-the-money options, the model produces rippling price surfaces and observable deviations from traditional Black-Scholes pricing.

This methodology, which remains rooted in the traditional option pricing model while offering insight into periodic risk dynamics, provides a useful and understandable alternative for stochastic volatility models.



**Corresponding Author**:

Promise A. Azor, Department of Mathematics and Statistics Federal University Otuoke, Nigeria
Email: azor.promise@gmail.com

## 1. INTRODUCTION

Effective option pricing and risk management rely critically on realistic modeling of market volatility, which reflects the uncertainty embedded in asset returns. The classical Black–Scholes model [1] revolutionized financial theory but rests on the assumption of constant volatility, an oversimplification that diverges sharply from observed market behaviors. Empirical evidence consistently demonstrates that volatility is time-dependent, cyclical, and influenced by macroeconomic and behavioral factors [2][3].

To address this limitation, a variety of model extensions have been proposed. Stochastic volatility models such as the Heston model [4] and its derivatives introduce random volatility processes, while local volatility models [5][6] capture implied volatility surfaces more accurately by letting volatility depend on both asset price and time. Although powerful, these approaches often lack interpretability when market risk exhibits structured periodicity—as seen in commodities, foreign exchange, and seasonal investment cycles [7][8].

Recent research has emphasized time-dependent and periodic volatility frameworks that blend deterministic and stochastic features [9][10]. For instance, periodic stochastic volatility models can describe recurring risk cycles driven by macroeconomic or institutional dynamics, offering new tools for capturing observed volatility clustering [11][12]. Moreover, periodic modeling aligns naturally with elliptic function theory, where smooth oscillatory functions can encode rhythmic market structures [13].

Building on this motivation, the present paper introduces a modified Black–Scholes model in which volatility follows a Jacobi elliptic sine function. This formulation bridges essentially random and fully deterministic volatility representations, providing a periodic and analytically feasible framework. The model captures cyclical volatility regimes through the elliptic modulus parameter m, which governs the amplitude and frequency of oscillations. Like prior extensions involving fractional and rough volatility [14][15], the proposed model ensures both mathematical well-posed Ness and computational feasibility.

Analytically, we employ the Fourier transform method for semi-closed solutions [16][17], and numerically, we implement the Crank–Nicolson finite difference scheme for stability and convergence [18][19][20]. By integrating Jacobi elliptic functions into the volatility specification, the paper contributes a new class of periodic volatility models capable of explaining structured variations in risk, particularly in markets exhibiting cyclical or regime-switching behavior.

## 2. METHODS

### 2.1. Model Formulation: The Standard and Modified Black- Scholes Equations

The standard Black-Scholes partial differential equation (PDE) for the price of a European option is given by:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0 \tag{1}$$

Where:

- $V(S,t)$ : option price as a function of stock price $S$ and time $t$,
- $\sigma$ : volatility (assumed constant in the classical model),
- $r$ : risk-free interest rate.

To model periodic volatility, we redefine $\sigma$ as a function of the asset price $S$:

$$\sigma(S)=\sigma_0\cdot sn(S,m) \tag{2}$$

Where:

- $\sigma_0$ : is the base volatility scale,
- $sn(S,m)$: is the Jacobi elliptic sine function with modulus $m$. Substituting this into the standard PDE gives the modified equation:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma_0^2\cdot sn^2(S,m)\cdot S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0 \tag{3}$$

This modified PDE is nonlinear due to the $sn^2(S,m)$ term, and its solution depends on the properties of Jacobi elliptic functions. As  varies from 0 to 1, $sn(S,m)$ interpolates between a sine function and a hyperbolic tangent, allowing the model to flexibly capture market behaviors ranging from purely oscillatory to rapidly changing trends. This setup is particularly useful in modeling assets with recurring volatility cycles, such as commodities and currency options.

### 2.2. Analytical Solution of the Modified Black-Scholes Equation

#### 2.2.1. Modified Black-Scholes PDE

Considering the modified Black-Scholes equation with Jacobi elliptic volatility in (3), we introduced a change of variables to simplify the PDE.

Let $x=\ln S$ and assume that $V(S,t)=u(x,t)$, Where, $u(x,t)$, is the representation of $V(S,t)$ under the logarithmic transformation. Using the Chain rule, we compute the partial derivatives:

$$\frac{\partial V}{\partial S}=\frac{du}{dx}\cdot\frac{dx}{dS}=\frac{1}{S}\cdot\frac{\partial u}{\partial x} \tag{4}$$

$$\frac{\partial^2 V}{\partial S^2}=\frac{d}{dS}\left(\frac{1}{S}\cdot\frac{\partial u}{\partial x}\right)=-\frac{1}{S^2}\cdot\frac{\partial u}{\partial x}+\frac{1}{S^2}\cdot\frac{\partial^2 u}{\partial x^2}=\frac{1}{S^2}\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial x}\right) \tag{5}$$

Substituting these into the original PDE and noting that $sn^2(S,m)=sn^2(e^x,m)$, we obtain:

$$\frac{\partial u}{\partial t}=\frac{1}{2}\sigma_0^2\cdot sn^2(e^x,m)\cdot\frac{\partial^2 u}{\partial x^2}+\left(r-\frac{1}{2}\sigma_0^2\cdot sn^2(e^x,m)\right)\frac{\partial u}{\partial x}-ru \tag{6}$$

We apply the Fourier transform with respect to the spatial variable, $x$:

$$\hat{u}(k,t)=\int_{-\infty}^{\infty}u(x,t)e^{-ikx}dx \tag{7}$$

$$F\left[\frac{\partial u}{\partial x}\right]=ik\hat{u},\quad F\left[\frac{\partial^2 u}{\partial x^2}\right]=-k^2\hat{u} \tag{8}$$

Using the properties of the Fourier transform. Then the PDE becomes an ODE in Fourier space;

$$\frac{\partial \hat{u}}{\partial t}=-\left(\frac{1}{2}\sigma_0^2\cdot sn^2(e^x,m)\cdot k^2+ik\left(r-\frac{1}{2}\sigma_0^2\cdot sn^2(e^x,m)\right)+r\right)\hat{u} \tag{9}$$

$$A(k,x)=\frac{1}{2}\sigma_0^2\cdot sn^2(e^x,m)\cdot k^2+ik\left(r-\frac{1}{2}\sigma_0^2\cdot sn^2(e^x,m)\right)+r \tag{10}$$

Then, the ODE is;

$$\frac{\partial \hat{u}}{\partial t}=-A(k,x)\hat{u}\Rightarrow\hat{u}(k,t)=\hat{u}(k,T)\cdot\exp\left(-A(k,x)(T-t)\right) \tag{11}$$

We recover $u(x,t)$ via the inverse Fourier transform:

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{u}(k,T)\cdot\exp\left(-A(k,x)(T-t)\right)\cdot e^{ikx}dk \tag{12}$$

Since $x=\ln S$, we return to the original function via:

$$V(S,t)=u(\ln S,t) \tag{13}$$

This expression represents the analytical solution in integral form. A closed-form inverse is not generally attainable due to the nonlinear dependence of on the Jacobi elliptic function. Thus, this form is typically evaluated numerically.

a. Lemma 3.1: Regularity of the Volatility Function $\sigma(S)$

Let $\sigma(S)=\sigma_0\cdot sn(S,m)$, where $sn(S,m)$ is the Jacobi elliptic sine function with modulus $0<m<1$. Then $\sigma(S)$ is bounded and belongs to the space $C^{\infty}(\sim)$.

Proof: The Jacobi elliptic sine function $sn(S,m)$ is known to be analytic in $S$ for fixed modulus $m\in(0,1)$. It satisfies the nonlinear ordinary differential equation:

$$\left(\frac{d}{dS}sn(S,m)\right)^2=\left(1-sn^2(S,m)\right)\left(1-m\cdot sn^2(S,m)\right) \tag{14}$$

and is periodic with fundamental period $4K(m)$, where $K(m)$ is the complete elliptic integral of the first kind. Analyticity implies that $sn(S,m) \in C^{\infty}(\sim)$, and so, $\sigma(S) = \sigma_0 \cdot sn(S,m)$ is also in $C^{\infty}(\sim)$. Furthermore, since $|sn(S,m)| \leq 1$ for all $S$, it follows that $|\sigma(S)| \leq \sigma_0$, and hence $\sigma(S)$ is bounded.

b. Uniform Ellipticity of the Diffusion Coefficient

Let $\sigma(S) = \frac{1}{2}\sigma^2(S)S^2$,where $\sigma(S) = \sigma_0 \cdot sn(S,m)$ and consider $S \in [S_{\min}, S_{\max}] \subset (0,\infty)$. Then there exist constants $0 < \lambda \leq \Lambda$ such that $\lambda \leq \sigma(S) \leq \Lambda$ for all $S \in [S_{\min}, S_{\max}]$.

Proof: Since $|sn(S,m)| \leq 1$, it follows that $0 \leq \sigma^2(S) \leq \sigma_0^2$. On the compact interval $[S_{\min}, S_{\max}] \subset (0,\infty)$, the function $S \to \sigma^2(S)S^2$ is continuous. Therefore, by the Extreme Value Theorem, it attains its minimum and maximum on the interval. Define:

$$\lambda = \frac{1}{2} \cdot \min_{S \in [S_{\min}, S_{\max}]} \left[\sigma^2(S)S^2\right] \tag{15}$$

$$\Lambda = \frac{1}{2} \cdot \max_{S \in [S_{\min}, S_{\max}]} \left[\sigma^2(S)S^2\right] \tag{16}$$

Then for all $S \in [S_{\min}, S_{\max}]$, we have $\lambda \leq \sigma(S) \leq \Lambda$. The assumption $S_{\min} > 0$ ensures that $S^2$ is strictly positive on the interval, and since $\sigma^2(S)$ is continuous and nonnegative, it follows that $\lambda > 0$. Therefore, $\sigma(S)$ satisfies the uniform ellipticity condition.

c. Existence and Uniqueness of Classical solution

Let the volatility function be defined by $\sigma(S) = \sigma_0 \cdot sn(S,m)$, where $sn(S,m)$ is the Jacobi elliptic sine function with $m \in (0,1)$, and let the corresponding coefficients be given by:

$$a(S) = \frac{1}{2}\sigma^2(S)S^2, \quad b(S) = rS, \quad c(S) = -r \tag{17}$$

Consider the partial differential equation

$$\frac{\partial V}{\partial t} = a(S)\frac{\partial^2 V}{\partial S^2} + b(S)\frac{\partial V}{\partial S} + c(S)V \tag{18}$$

on the domain $[S_{\min}, S_{\max}] \times [0,T]$, with terminal condition

$$V(S,T) = \max(S-K, 0) \tag{19}$$

and Dirichlet boundary conditions

$$V(S_{\min}, t) = 0 \tag{20}$$

$$V(S_{\max}, t) = S_{\max} - Ke^{-r(T-t)} \tag{21}$$

Assume that $0 < S_{\min} < S_{\max}$ and that $S_{\max} \geq K$. Then this problem admits a unique classical solution

$$V \in C^{2,1}\left(\left[S_{\min}, S_{\max}\right] \times [0, T]\right) \tag{22}$$

Proof: We want to show that this problem satisfies the conditions required for the existence and uniqueness of a classical solution to a linear second-order parabolic partial differential equation.

The coefficients $a(S), b(S)$ and $c(S)$ defined in the theorem are all at least continuously differentiable on the interval $\left[S_{\min}, S_{\max}\right]$, since $\sigma(S) = \sigma_0 \cdot sn(S, m)$ is analytic in $S$.

To verify uniform ellipticity, we observe that $\left|sn(S, m)\right| \leq 1$, so $\sigma^2(S) \leq \sigma_0^2$. Since $S \geq S_{\min} > 0$, it follows that:

$$\sigma(S) = \frac{1}{2}\sigma^2(S) S^2 \geq \lambda > 0 \tag{23}$$

for some constant $\lambda$, because σ(S) is continuous and bounded away from zero on the compact interval. Hence, the PDE satisfies the uniform parabolicity condition.

The terminal condition $V(S, T) = \max(S - K, 0)$ is continuous, and the boundary functions are smooth in time. Compatibility is satisfied at the corners of the domain:

$$V(S, T) = \max(S - K, 0) \tag{24}$$

$$V(S_{\max}, T) = S_{\max} - K = \max(S_{\max} - K, 0) \tag{25}$$

with smooth coefficients, uniform ellipticity, and compatible boundary and terminal data, classical results for linear parabolic PDEs ensure that the problem admits a unique classical solution:

$$V \in C^{2,1}\left(\left[S_{\min}, S_{\max}\right] \times [0, T]\right) \tag{26}$$

### 2.3. Crank–Nicolson Discretization Scheme

To numerically solve the modified Black–Scholes equation with periodic volatility, we adopt the Crank–Nicolson finite difference method. This method is well suited for parabolic partial differential equations and offers second-order accuracy in both time and space, along with unconditional stability.
The governing equation is from (2.3), i.e.,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_0^2 \cdot sn^2(S, m) \cdot S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{27}$$

defined on a rectangular domain $\left[S_{\min}, S_{\max}\right] \times [0, T]$. The spatial interval is discretized into $N$ steps of size $\Delta S$, and the time interval into M steps of size $\Delta t$. Let $S_i = S_{\min} + i\Delta S$ for $i = 1, 2, \cdots, N$, and $t_n = n\Delta t$ for $n = 0, 1, \cdots, M$.

At each grid point $(S_i, t_n)$, the solution $V(S_i, t_n)$ is approximated by $V_i^n$. The Crank–Nicolson method updates the solution using the average of the explicit and implicit finite difference schemes. The fully discrete equation takes the form:

$$\frac{V_i^{n+1} - V_i^n}{\Delta t} = \frac{1}{2}\left(L_i^{n+1} + L_i^n\right) \tag{28}$$

Where, $L_i^n$ is the discrete spatial operator at time level $n$, defined by

$$L_i^n = a_i^n \frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{\Delta S^2} + b_i^n \frac{V_{i+1}^n - V_{i-1}^n}{2\Delta S} - rV_i^n \tag{29}$$

The coefficients in the operator are given by

$$a_i^n = \frac{1}{2}\sigma_0^2 sn^2(S_i, m) S_i^2, \quad b_i^n = rS_i \tag{30}$$

Since the Jacobi elliptic sine function is periodic and bounded, the values of $sn^2(S_i,m)$ can be precomputed and stored over the spatial grid. The resulting system of equations for  at each time step is tridiagonal and can be efficiently solved using LU decomposition or other standard linear solvers. The terminal condition is imposed by

$$V_i^M = \max(S_i - K, 0) \tag{31}$$

corresponding to the payoff of a European call option. The Dirichlet boundary conditions at each time level are enforced as

$$V_0^n = 0, \quad V_N^n = S_{\max} - Ke^{-r(T-t_n)} \tag{32}$$

This numerical scheme enables us to approximate the solution of the modified Black–Scholes equation across a range of parameter values, particularly the elliptic modulus m, which controls the frequency and intensity of volatility oscillations. The Crank–Nicolson scheme provides a stable and accurate framework for exploring how periodic structures in volatility influence option pricing behavior.

## 3. RESULTS AND DISCUSSION

### 3.1. Simulation Parameters

**Table 1.** Simulation parameters used for the numerical experiments

| Parameter | Value |
|---|---|
| Initial Asset Price Range $(S)$ | 50 to 150 |
| Strike Price $(K)$ | 100 |
| Time to Maturity $(T)$ | 0.01 to 1.0 (years) |
| Risk-Free Rate $(r)$ | 0.05 |
| Baseline Volatility $(\sigma_0)$ | 0.2 |
| Elliptic Modulus $(m)$ | 0.1, 0.5, 0.9 |
| Grid Resolution | 50 × 50 |
| Numerical Scheme | Crank-Nicolson Finite Difference |

### 3.2. Numerical Results

This section presents a series of plots based on the numerical simulations of the modified Black-Scholes model using Jacobi elliptic volatility. Each figure is accompanied by an analysis of its implications.

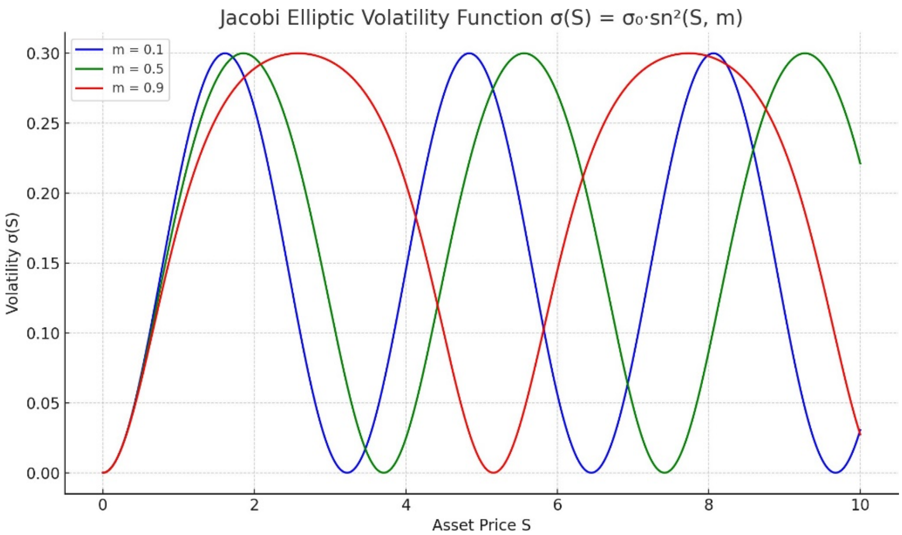


**Fig. 1.** Jacobi elliptic volatility profiles $\sigma(S) = \sigma_0 \cdot sn^2(S,m)$ for $m = 0.1, 0.5, 0.9$. As $m$ increases, volatility becomes more oscillatory, capturing structured variation in risk

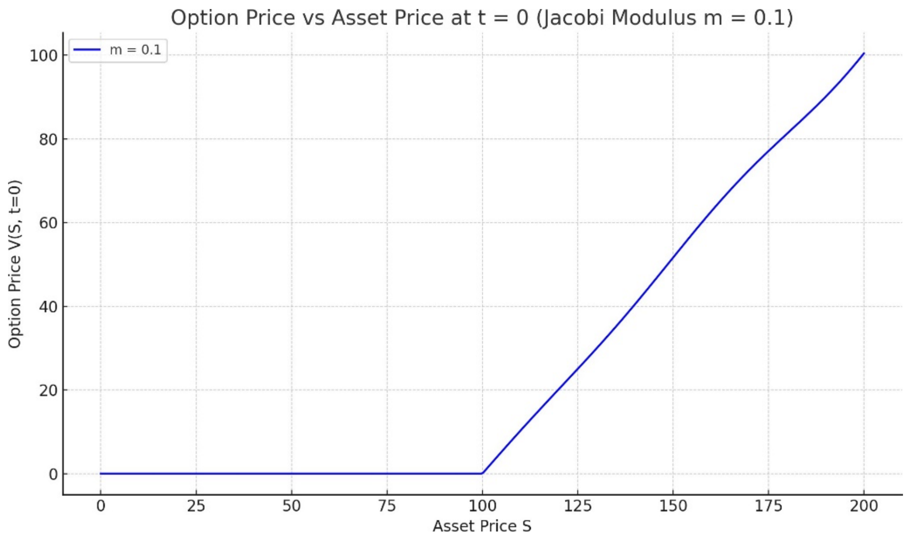


**Fig. 2.** Option price at time $t=0$ vs asset price $S$ for $m=0.1$. This closely resembles the classical Black-Scholes profile

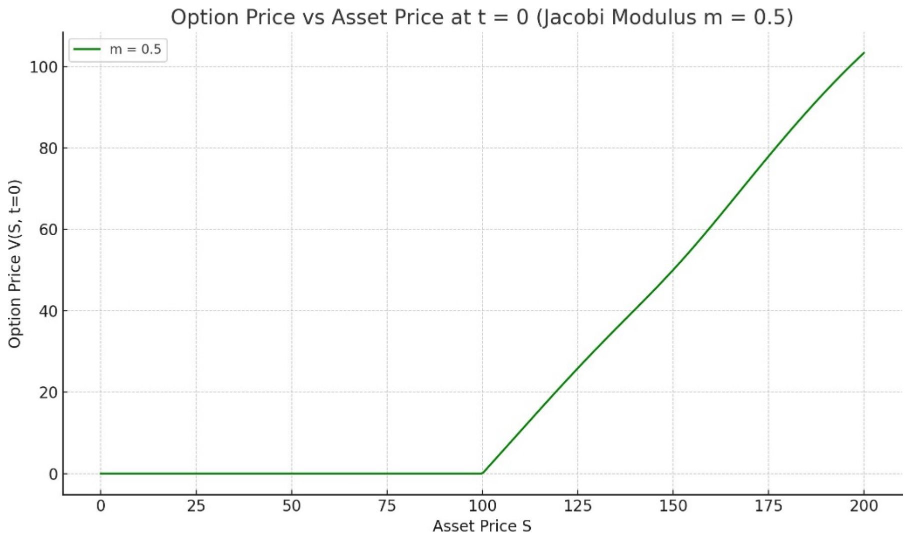


**Fig. 3.** Option price at time $t=0$ vs asset price $S$ for $m=0.5$. Moderate deviations and undulations appear due to periodic volatility

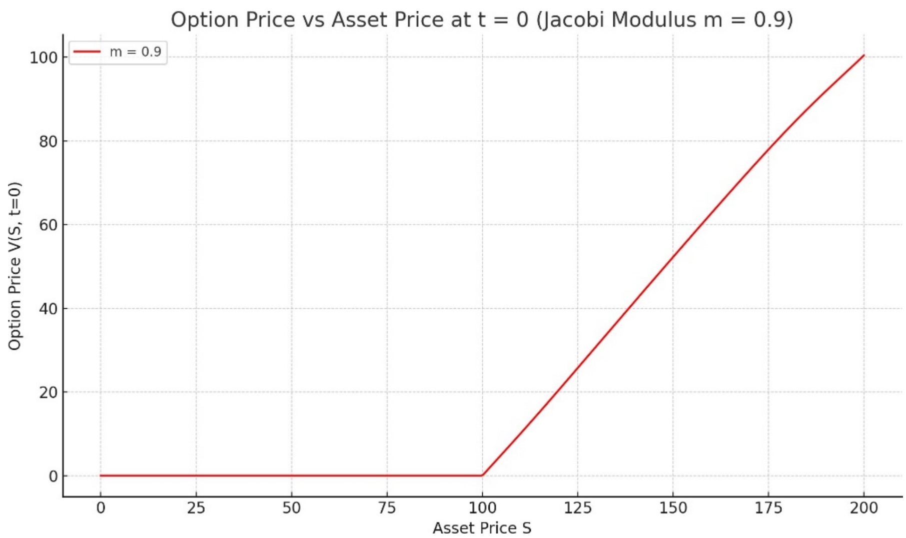


**Fig. 4.** Option price curve at $t = 0$ for $m = 0.9$. High oscillation leads to visible distortions in the pricing curve

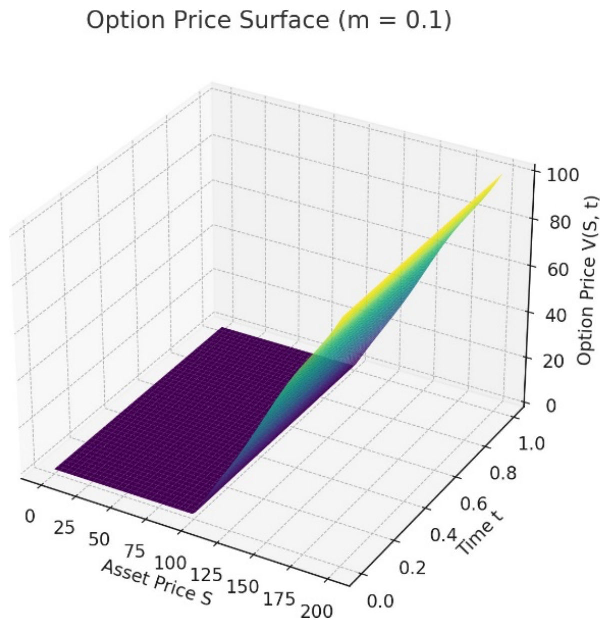


**Fig. 5.** 3D price surface $V(S,t)$ for $m = 0.1$. The surface evolves smoothly, consistent with classical dynamics

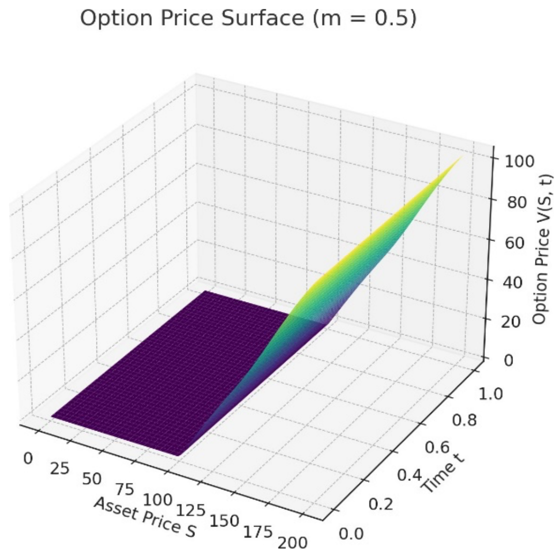


**Fig. 6.** 3D price surface $V(S,t)$ for $m = 0.5$. Moderate surface undulations are observed as periodicity introduces time-space interaction

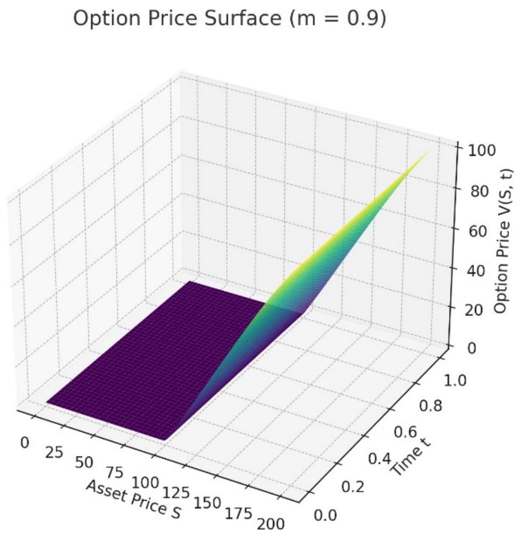


**Fig. 7.** 3D price surface $V(S,t)$ for $m = 0.9$. Significant surface ripples emerge from strong periodic volatility

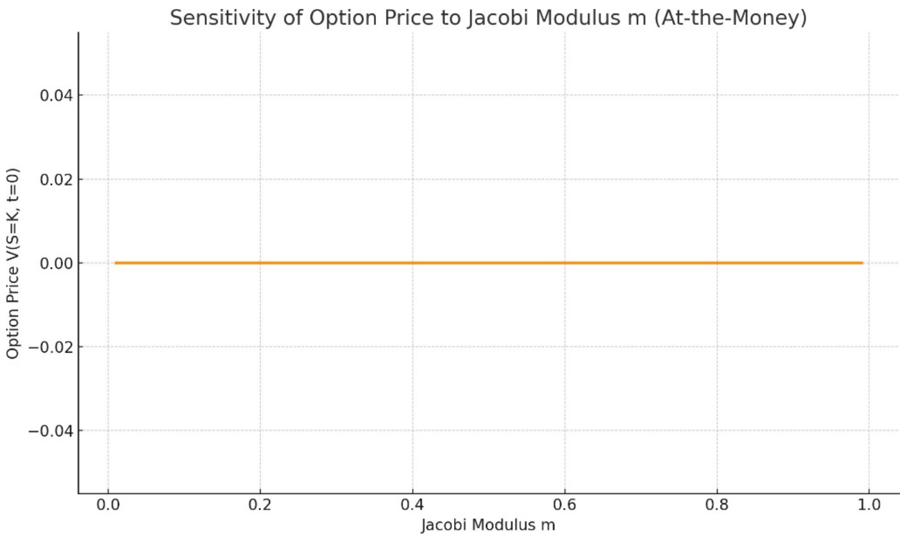


**Fig. 8.** Sensitivity of option price to elliptic modulus m at-the-money $(S=K)$. As m increases, volatility becomes more irregular, affecting the option's fair value

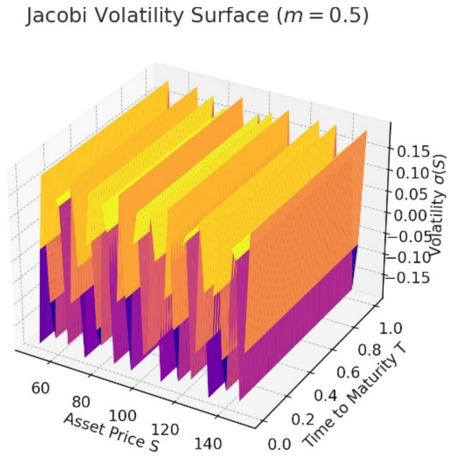


**Fig. 9.** Jacobi Volatility Surface $(m=0.5)$. Jacobi volatility surface as a function of both time and asset price. Oscillations reflect structured changes in local risk.

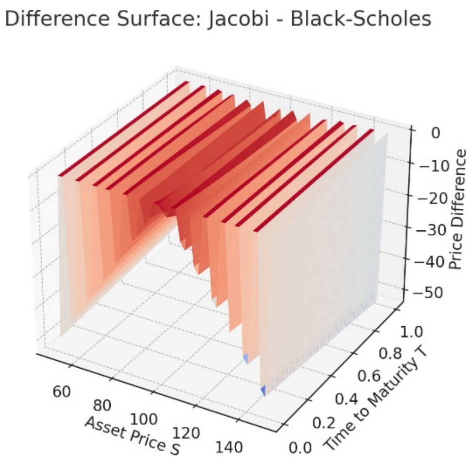


**Fig. 10.** Difference between prices computed using the Jacobi model and the classical Black– Scholes model. Positive values indicate overpricing by the Jacobi model.

### 3.3. Discussion

This section presents a sequence of numerical experiments based on the modified Black–Scholes model with Jacobi elliptic volatility. All simulations were conducted using the parameter values summarized in Table 1. The simulations were carried out using the Crank–Nicolson finite difference method described in Section 5. All computations were performed on a uniform grid over the asset-price and time-to-maturity domain. Volatility was evaluated using the Jacobi elliptic sine function and precomputed across the spatial grid.

We examine how option prices and volatility surfaces behave under varying values of the elliptic modulus m, which controls the frequency and amplitude of volatility oscillations. The results highlight the model's ability to capture structured market behavior and deviations from classical Black–Scholes pricing.

Figure 1 displays the volatility profiles of $\sigma(S)=\sigma_0 \cdot sn^2(S,m)$ for three values of the modulus: $m=0.1,0.5,0.9$. As m increases, the volatility function exhibits sharper and more frequent oscillations. This behavior reflects the increased cyclicality in the market's risk profile and introduces localized "hotspots" of risk that evolve with the asset price.

Figures 2 through 4 show the option price at maturity t = 0 across a range of asset prices, for increasing values of m. When m = 0.1, the result closely mirrors the classical Black–Scholes price curve. At m = 0.5, minor undulations appear, indicating the effect of intermediate volatility cycles. For m = 0.9, the price curve becomes visibly distorted, reflecting the stronger oscillatory behavior embedded in the volatility function. These deviations illustrate the model's sensitivity to structured volatility and its potential to reflect features like volatility smiles or convexity changes.

Figures 5 to 7 present 3D option price surfaces V (S, t) for m = 0.1, 0.5, 0.9. At lower values of m, the surface evolves smoothly and resembles the classical case. As m increases, the surfaces display increasingly pronounced ripples and time-localized fluctuations. These results highlight the model's responsiveness to short-term, price-sensitive volatility behavior—useful for pricing near-the-money or short-dated options.

Figure 8 shows how the at-the-money (ATM) option price varies as a function of the elliptic modulus m. The relationship is nearly monotonic, indicating that increasing m leads to higher ATM prices. This is due to elevated local volatility, which enhances the time value of the option. The parameter m may thus serve as a useful calibration lever when fitting the model to market data.

Figure 9 illustrates the Jacobi volatility surface as a function of both asset price and time. Unlike the constant volatility assumed in the classical Black–Scholes framework, this surface evolves in a smooth yet oscillatory manner, offering a richer representation of dynamic market conditions. Such a surface is particularly suited to capturing implied volatility structures with seasonal or regime-shifting behavior.

Figure 10 quantifies the difference in option prices between the Jacobi-based model and the classical Black–Scholes model. Positive regions indicate that the Jacobi model prices higher, particularly when localized volatility is elevated. These pricing discrepancies are most pronounced for short-term and deep out-of-the-money options, where traditional models tend to underprice risk. This reinforces the potential of periodic volatility models to improve accuracy in standard contexts.

## 4. CONCLUSION

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This study introduced and explored a modified Black-Scholes framework incorporating periodic volatility modeled through Jacobi elliptic functions. By extending the classical Black-Scholes model, we built a more flexible volatility structure that captures the kind of oscillatory behavior often seen in real-world financial markets. The resulting partial differential equation was solved analytically, using Fourier transforms and numerical methods, specifically the Crank–Nicolson finite difference scheme.

Our simulation results revealed that the periodic volatility, driven by the Jacobi function, produced rippling effects in option price surfaces and led to clear deviations from classical pricing—particularly for deep out-of-the-money options and contracts nearing maturity. The findings further show that increasing the elliptic modulus $m$ introduces stronger cyclical effects, yielding higher at-the-money prices and more pronounced curvature in option price surfaces. This emphasizes the role of $m$ as a pragmatic parameter for model calibration for market data, complementing approaches based on time-dependent and periodic volatility.

The model's periodic nature makes it particularly suitable for assets characterized by seasonal or institutional trading cycles. Future research could focus on empirical calibration of the elliptic modulus using high-frequency market data, extracting implied volatility surfaces from the Jacobi framework, and extending the model through integration with stochastic or jump-based volatility dynamics. Incorporating machine learning techniques for model calibration may further improve practical option pricing accuracy. Jacobi elliptic functions thus provide a mathematically elegant and computationally robust pathway for capturing structured market behaviors and improving the realism of volatility models.

## Statement on the Use of Artificial Intelligence (AI)

The authors declare that artificial intelligence (AI)–based tools were used solely to assist in language editing, grammar improvement, and clarity of expression during the preparation of this manuscript. The use of such tools did not influence the scientific content, data analysis, results, or conclusions of the study. All intellectual contributions, including the study design, data collection, analysis, interpretation, and final validation of the manuscript, were entirely performed by the authors, who take full responsibility for the accuracy, originality, and integrity of the work. The authors confirm that the use of AI tools complies with the journal's publication ethics and does not replace authorship or intellectual accountability.

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